MATH 1650: SECTION 2.4: COMPLEX ZEROS OF POLYNOMIALS

In the previous section, we restricted our attention to finding the **real** zeros of a polynomial function, p. The real zeros of p correspond to the x-intercepts of the graph of $y = p(x)$. In this section, we are also concerned with finding the non-real zeros of a polynomial function and then using all the zeros to completely factor $p(x)$.

The big results that drive what we do in this section are below:

THE COMPLEX CONJUGATE PAIRS THEOREM: For polynomial functions with real number coefficients, non-real zeros come in complex conjugate pairs: $a + bi$ and $a - bi$.

THE REAL FACTORIZATION THEOREM: Every polynomial function with real number coefficients can be factored into a product of (repeated) linear factors which correspond to the real zeros of p and irreducible quadratic factors which correspond to the non-real zeros of p.

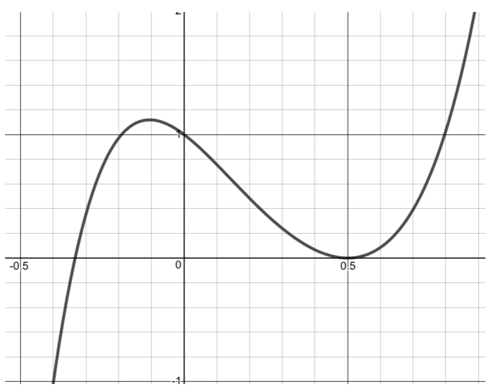
THE COMPLEX FACTORIZATION THEOREM: Every polynomial function can be completely completely factored over the complex numbers as follows:

$$p(x) = (\text{leading coefficient})(x - \text{each zero})^{\text{corresponding multiplicity}}$$

EXAMPLE: Let $f(x) = 12x^5 - 20x^4 + 19x^3 - 6x^2 - 2x + 1$.

- Find the real zeros of f and factor $f(x)$ over the real numbers.

  Since f is a fifth degree polynomial, we know that we need to perform at least three successful divisions to get the quotient down to a quadratic function. Using desmos, we create the graph below on the left. The RZT suggests $x = \frac{1}{2}$ and $x = -\frac{1}{3}$ as zeros, which we verify using synthetic division below on the right.



$$\begin{array}{r|rrrrrr}
\frac{1}{2} & 12 & -20 & 19 & -6 & -2 & 1 \\
 & \downarrow & 6 & -7 & 6 & 0 & -1 \\
\hline
\frac{1}{2} & 12 & -14 & 12 & 0 & -2 & \boxed{0} \\
 & \downarrow & 6 & -4 & 4 & 2 & \\
\hline
-\frac{1}{3} & 12 & -8 & 8 & 4 & \boxed{0} & \\
 & \downarrow & -4 & 4 & -4 & & \\
\hline
 & 12 & -12 & 12 & \boxed{0} & &
\end{array}$$

  Our quotient is $12x^2 - 12x + 12$. To solve $12x^2 - 12x + 12 = 0$, we first factor: $12(x^2 - x + 1) = 0$ and set $x^2 - x + 1 = 0$. Since this quadratic does not factor nicely, we appeal to the quadratic formula. We check the discriminant: $b^2 - 4ac = (-1)^2 - 4(1)(1) = -3 < 0$ which means the remaining zeros of f are non-real. Hence, we use our synthetic division tableau to factor $f(x)$ over the real numbers as:

$$f(x) = \left(x - \frac{1}{2}\right)^2 \left(x + \frac{1}{3}\right) \left(12x^2 - 12x + 12\right) = 12 \left(x - \frac{1}{2}\right)^2 \left(x + \frac{1}{3}\right) \left(x^2 - x + 1\right)$$

- Find the non-real zeros of f and factor $f(x)$ over the complex numbers.

  Using the Quadratic Formula, we solve $x^2 - x + 1 = 0$: $x = \dfrac{1 \pm \sqrt{1 - 4(1)(1)}}{2(1)} = \dfrac{1 \pm \sqrt{-3}}{2} = \dfrac{1 \pm i\sqrt{3}}{2}$.

  Applying the Complex Factorization Theorem we factor $f(x)$ as:

$$f(x) = 12 \left(x - \frac{1}{2}\right)^2 \left(x + \frac{1}{3}\right) \left(x - \left[\frac{1 + i\sqrt{3}}{2}\right]\right) \left(x - \left[\frac{1 - i\sqrt{3}}{2}\right]\right)$$

EXAMPLE: Let $p(x) = 2x^5 + 5x^4 + 5x^3 + 2x^2 - x - 1$.

- Find the real zeros of p and factor $p(x)$ over the real numbers.

- Find the non-real zeros of p and factor $p(x)$ over the complex numbers.

EXAMPLE: Let $p(z) = z^4 - 3z^3 + 5z^2 - 12z + 4$.

- Show p has no rational zeros.

- Use synthetic division to show $z = 2i$ is a zero of p and use this to find the remaining complex zeros of p.

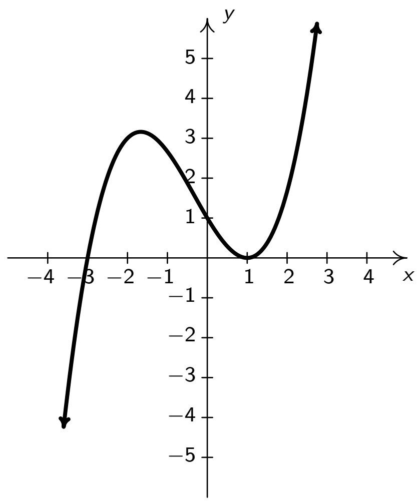
  HINT: Once you show $z = 2i$ is a zero, what else must be a zero?

- Factor $p(z)$ over the complex numbers.

- Factor $p(z)$ over the real numbers.

EXAMPLE: Find a possible formula for the polynomial functions described below:

- the polynomial function graphed below:



$y = f(x)$

- $p(t)$ where:
  - p has real number coefficients.
  - p is degree 4.
  - $t = 2 - i$ is zero.
  - the point $(-3, 0)$ is a local maximum.

HOMEWORK: Section 2.4: 1 - 37 odd